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Rational approximation in uniform and weighted L^1 norm

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Abstract

The authors study the rational approximation of functions nonuniformly smooth. New uniform and weighted mean error estimates are given.

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1. Introduction

In the present paper we want to investigate the rational approximation of functions not having a homogeneous behaviour on $[-1,1]$. More precisely, we want to consider the class $A = A(p,q,a)$ of all functions satisfying the following properties:

(i) $f \in C^p([-1,1])$, $p \geqslant 1$;
(ii) $f \in C^q([c,d])$, with $p < q < \infty$, for every closed segment $[c,d] \subset [-1,1] - \{a\}$, $|a| \leqslant 1$;
(iii) there exists $\lim_{x\to a, x\neq a}(a-x)^k f^{(p+k)}(x) < \infty$, $|x| \leqslant 1$, $k = 1,\ldots,q-p$.

Examples of such functions are

$$f(x) = \begin{cases} (1+x)^{3/2}, & -1 \leqslant x \leqslant 0, \\ 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \frac{3}{128}x^4 - \frac{3}{256}x^5, & 0 \leqslant x \leqslant 1, \end{cases}$$

belonging to $C^1([-1,1])$ and $C^5((-1,1))$ with the largest value of q being 5; and

$$g(x) = \begin{cases} (-x)^{9/2} + (1+x)^{9/2}, & -1 \leqslant x \leqslant 0, \\ x^{4/3} + (1+x)^{9/2}, & 0 \leqslant x \leqslant 1, \end{cases}$$

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belonging to $C^1([-1,1])$ and $C^4([-1,1]-\{0\})$, with the largest value of q being 4.

In the case $a=\pm1$, the functions of the class A are connected with the solutions of integral equations with singular kernels (see [14]).

Many authors considered the polynomial and rational approximation of analytic functions with singularities, i.e., when $q=\infty$; among all we mention [9,10,13,15,19,24].

The polynomial approximation of functions of the class A, when $a=\pm1$, was studied in [14]. As far as we know, results about rational approximation of the above class do not exist.

Theorem 2.1, proved in the present paper, shows the behaviour of best uniform rational approximation of functions $f \in A$; the estimate (2.1) is new.

In this paper we want also to consider the class B, formed by functions f having in the point $a \in [-1,1]$ an integrable singularity and in addition $f \in C^q([c,d])$, for every closed segment $[c,d] \subset [-1,1]-\{a\}$. For the functions of the above class we study the rational approximation in the L^1 norm, with respect to a weight function having an algebraic singularity in the range $[-1,1]$. The results are shown in Theorem 2.2.

Finally we note that, to prove Theorems 2.1 and 2.2, we demonstrate some lemmas on pointwise rational approximation estimates; those results can be used in different contexts.

2. Main results

The first result regards the rational approximation of functions $f \in A$. For such functions we can define a new function $\phi(x)=(a-x)^{q-p}f(x)$, with the qth derivative continuous, by (iii) ($\phi \in C^q([-1,1])$).

Then, letting $R_{r,s}(g)=\inf\{\|g-R\|,\ R \in \mathcal{R}_{r,s}\}$, where $g \in C^0([-1,1])$, $\|\cdot\|$ denotes the uniform norm and $\mathcal{R}_{r,s}$ is the set of rational functions of degree (r,s), we can state the following theorem.

Theorem 2.1. *Let $f \in A(1,q,a)$, $q>2$. Then,*

$$R_{n+p,n}(f) \leqslant C\frac{R_{n-q,n}(\phi^{(q)})}{n^{q-\epsilon}}, \quad \forall\epsilon>0, \tag{2.1}$$

with n (even) $>q$, $\phi(x)=(a-x)^{q-p}$ and C a positive constant independent of f and n.

Remarks. If $f^{(p+1)}$ is integrable, the estimate

$$R_{n,n}(f) \leqslant C\frac{\|f^{(p+1)}\|_{L^1}}{n^{p+1}}, \quad n \geqslant p,$$

is known [18]. Therefore, if $f \in A(p,q,a)$, $q>2$, by the previous estimate we get

$$R_{n,n}(f)=\mathrm{O}(n^{-p-1}).$$

On the other hand, from Theorem 2.1, since $\phi \in C^q([-1,1])$ by (iii), there follows

$$R_{n+p,n}(f) \leqslant \mathrm{o}(n^{-q+\epsilon}), \quad \forall\epsilon>0.$$

Moreover, estimate (2.1) is not possible by polynomial approximation. In fact, if $|a|<1$ and $q=\infty$, the best uniform approximation error is $\mathrm{O}(n^{-p})$ (see, for example, [15]); when $a=\pm1$, then the best uniform approximation error is $\mathrm{O}(n^{-\gamma})$, with $\gamma=\min(q,2p)$ and $p<q$ (see [14,24]).

Now we want to state a result for functions $f \in B$. Analogously we define the function $\psi(x) = (x-a)^{q+1} f(x)$ and from the definition it follows that $\psi \in C^q([-1,1])$.

Theorem 2.2. *If $\psi^{(q)} \in \mathrm{Lip}(1+\sigma)$, $q \geqslant 1$ and $-1 < \sigma < 0$, then for every n there exists a rational function $r_{n-1,n}$ of degree $(n-1,n)$ independent of y such that*

$$\int_{-1}^{1} |f(x) - r_{n-1,n}(x)|\, |x-y|^{\nu}\, \mathrm{d}x \leqslant C \frac{1}{n^A}, \quad |y| \leqslant 1, \ \forall A > \sigma + 1, \tag{2.2}$$

where $-1 < \nu < 0$, $\sigma + 1 + \nu > 0$ and $C > 0$ is a constant independent of f and n.

We remark that, in the case of polynomial approximation, an analogous theorem holds [14], but the right-hand side in (2.2) is replaced by $\mathrm{O}(1/n^{2\sigma+2\nu+2})$.

3. Proofs of Theorems 2.1 and 2.2

To prove Theorems 2.1 and 2.2, some results on rational approximation are needed.

Let $Y = (y_{m,i} \in [-1,1]\colon\ i = 0,\dots,m;\ m \in \mathbb{N})$ be an infinite matrix where each row is a set of distinct points in $[-1,1]$ and define for $f \in C^r([-1,1])$ the operator

$$S_{m,r}(Y;f;x) = S_{m,r}(f;x) = \frac{\sum_{k=0}^{m} \left(|x-y_{m,k}|^{-s} \sum_{j=0}^{r} f^{(j)}(y_{m,k})(x-y_{m,k})^j/(j!) \right)}{\sum_{k=0}^{m} |x-y_{m,k}|^{-s}},$$

with $s = r + \alpha$, where $\alpha > 2$ is a fixed integer number. It follows that $S_{m,r}$ is a rational linear operator of degree $(r + sm, sm)$. Moreover, it preserves polynomials of degree r, i.e.,

$$S_{m,r}(Y;e_i;x) = e_i, \quad e_i(x) = x^i, \quad i = 0,\dots,r,$$

$$D^i S_{m,r}(Y;f;y_{m,k}) = f^{(i)}(y_{m,k}), \quad k = 0,\dots,m, \quad i = 0,\dots,r,$$

and

$$D^i S_{m,r}(Y;f;y_{m,k}) = 0, \quad k = 0,\dots,m, \quad i = r+1,\dots,[s]-1.$$

This operator contains as particular cases the Szabados operator [22, p.220] and the Shepard–Balázs operator widely used in approximation theory and in fitting data, curves and surfaces [1–3,6–8,12,16,17,20,21,23].

Recently the authors [5] obtained, for some particular nodes matrices, pointwise simultaneous approximation estimates which are not possible by polynomial approximation.

Indeed, given a real number $\ell > \frac{1}{2}$, let $x = x_\ell : [0,1] \to [-1,1]$ be the function defined by

$$x = x(\theta) = \begin{cases} (2\theta)^{2\ell+1} - 1, & \theta \in [0, \frac{1}{2}], \\ -(2-2\theta)^{2\ell+1} + 1, & \theta \in [\frac{1}{2}, 1], \end{cases} \tag{3.1}$$

and consider the matrix $X = (x_{m,k} = x(k/m)\colon\ k = 0,\dots,m,\ m \in \mathbb{N})$. Then we have the following lemma [5].

Lemma 3.1. *For all $f \in C^r([-1,1])$ and for all $i = 0, \ldots, r$,*

$$|[f(x) - S_{m,r}(X;f;x)]^{(i)}| \leqslant C \frac{[(1-x^2)^{2\ell/(2\ell+1)}]^{r-i}}{m^{r-i}} \omega\left(f^{(r)}; \frac{(1-x^2)^{2\ell/(2\ell+1)}}{m}\right), \tag{3.2}$$

where $x \in [-1,1]$, $s = r + \alpha$, $\alpha > 2$ is a fixed integer, C is a constant depending only on ℓ, r and α and $\omega(g; \cdot)$ is the usual modulus of continuity of g.

We remark that estimates of type (3.2) are not possible by polynomial approximation, as Gopengauz [11] showed.

Moreover, the endpoints are not special points and we can get results analogous to (3.2) for any interior point. For example, setting for $\ell > \frac{1}{2}$,

$$z = z(\theta) = \begin{cases} -(1-2\theta)^{2\ell+1}, & \theta \in [0, \frac{1}{2}], \\ (2\theta - 1)^{2\ell+1}, & \theta \in [\frac{1}{2}, 1], \end{cases}$$

and $Z = (z_{m,k} = z(k/m)\colon\ k = 0, \ldots, m)$ with m even, we get the following lemma [5].

Lemma 3.2. *For all $f \in C^r([-1,1])$ and for all $i = 0, \ldots, r$,*

$$|[f(x) - S_{m,r}(Z;f;x)]^{(i)}| \leqslant C \frac{[|x|^{2\ell/(2\ell+1)}]^{r-i}}{m^{r-i}} \omega\left(f^{(r)}; \frac{|x|^{2\ell/(2\ell+1)}}{m}\right), \tag{3.3}$$

where $x \in [-1,1]$, $s = r + \alpha$, $\alpha > 2$ is a fixed integer and C is a constant depending only on ℓ, r and α.

Without loss of generality, assume now $a = 0$. Then, letting $\phi(x) = x^{q-p} f(x)$, the following lemma holds.

Lemma 3.3. *There exists a sequence of rational functions $r_{n+p,n}(x)$ of degree $(n+p, n)$, n (even) $> q$, such that*

$$|f(x) - r_{n+p,n}(x)| \leqslant C \frac{\omega(\phi^{(q)}; |x|^{2\ell/(2\ell+1)}/n)}{n^q |x|^{q/(2\ell+1)-p}}, \quad |x| > n^{-2\ell-1}, \tag{3.4}$$

$$|f(x) - r_{n+p,n}(x)| \leqslant C\omega\left(\phi^{(q)}; \frac{|x|^{2\ell/(2\ell+1)}}{n}\right) \frac{|x|^{p(2\ell)/(2\ell+1)}}{n^p}, \quad |x| \leqslant n^{-2\ell-1}, \tag{3.5}$$

where ℓ, p and q are arbitrary but fixed integers and C is a constant independent of f, n, x, but dependent of ℓ, p and q.

Remark 3.4. Lemma 3.3 says that we can find a sequence of rational operators $\{r_{n+p,n}\}_n$ approximating f well on $[-1,1]$ and converging to f faster away from 0.

The approximation of f by rational operators r_n is better where f is smoother. This shows once more that rational approximation is more sensible than polynomials about the smoothness of f (see also [18]).

Proof of Lemma 3.3. From the assumptions we know that $\phi(x) = x^{q-p}f(x) \in C^q([-1,1])$; hence, by Lemma 3.2, for m even,

$$|[\phi(x) - S_{m,q}(\phi;x)]^{(i)}| \leqslant A\frac{[|x|^{2\ell/(2\ell+1)}]^{q-i}}{m^{q-i}}\omega\left(\phi^{(q)};\frac{|x|^{2\ell/(2\ell+1)}}{m}\right), \quad i=0,\dots,q, \tag{3.6}$$

where $x \in [-1,1]$, $s = q+\alpha$, $\alpha > 1$ is a fixed integer and A is a constant depending only on ℓ, q and α.

Since $\phi^{(k)}(0) = 0$, $k = 0,\dots,q-p-1$, it follows that $(S_{m,q})^{(k)}(\phi;0) = 0$, $k = 0,\dots,q-p-1$. Therefore,

$$S_{m,q}(\phi;x) = x^{q-p}r_{n+p,n}(x), \quad n = m(q+\alpha),$$

with $r_{n+p,n}(x)$ a rational function of degree $(n+p,n)$. Inserting this latter expression into (3.6) with $i = 0$, we get (3.4).

To prove (3.5), from Taylor's formula for $\phi - S_{m,q}(\phi)$, we can write

$$|f(x) - r_{n+p,n}(x)| = \frac{|\phi(x) - S_{m,q}(\phi;x)|}{|x|^{q-p}} = \frac{|\phi^{(q-p)}(\xi_x) - S_{m,q}^{(q-p)}(\phi;\xi_x)|}{(q-p)!},$$

$0 < \xi_x < |x| \leqslant n^{-2\ell-1}$, and by (3.6), (3.5) follows. □

Proof of Theorem 2.1. Let g_n be the rational function of degree (n,n), n even, such that

$$\|\phi^{(q)} - g_n^{(q)}\| \leqslant R_{n-q,n}(\phi^{(q)}),$$

and let $\bar{g}_n(x) = g_n(x)x^{-q+p}$.

From Lemma 3.3, there exists a rational function $r_{n+p,n}$ of degree $(n+p,n)$, such that, for $|x| > n^{-2\ell-1}$,

$$\begin{aligned}|(f-\bar{g}_n)(x) - r_{n+p,n}(x)| &\leqslant \frac{\text{const.}}{q!}\frac{\omega((\phi-g_n)^{(q)};1/n)}{n^q|x|^{q/(2\ell+1)-p}} \leqslant \frac{\text{const.}}{q!}\frac{\|(\phi-g_n)^{(q)}\|}{n^q|x|^{q/(2\ell+1)-p}}\\ &\leqslant \frac{\text{const.}}{q!}\frac{R_{n-q,n}(\phi^{(q)})}{n^q|x|^{q/(2\ell+1)-p}},\end{aligned}$$

that is,

$$|f(x) - \bar{r}_{n+p,n}(x)| \leqslant C\frac{R_{n-q,n}(\phi^{(q)})}{n^q|x|^{q/(2\ell+1)-p}}, \quad |x| > n^{-2\ell-1},$$

with $\bar{r}_{n+p,n} = r_{n+p,n} + \bar{g}_n$.

Analogously, from Lemma 3.3, we can prove

$$|f(x) - \bar{r}_{n+p,n}(x)| \leqslant CR_{n-q,n}(\phi^{(q)})\frac{|x|^{p(2\ell)/(2\ell+1)}}{n^p}, \quad |x| \leqslant n^{-2\ell-1},$$

from which the assertion follows. □

Proof of Theorem 2.2. From (3.6), if $i = 0$, we get

$$|\psi(x) - S_{n,q}(\psi;x)| \leqslant C\frac{\omega(\psi^{(q)};|x|^{2\ell/(2\ell+1)}/n)}{|x|^{2\ell/(2\ell+1)}}\frac{|x|^{2\ell(q+1)/(2\ell+1)}}{n^q}, \quad n^{-2\ell-1} < |x|;$$

therefore,

$$|f(x) - r_{n-1,n}(x)| \leqslant C\frac{\omega(\psi^{(q)}; |x|^{2\ell/(2\ell+1)}/n)}{|x|^{2\ell/(2\ell+1)}/n}\frac{1}{n^{q+1}|x|^{(q+1)/(2\ell+1)}} \leqslant C\frac{|x|^{-(q+1)/(2\ell+1)}}{n^{q+1}n^{\sigma(2\ell+1)}}. \tag{3.7}$$

On the other hand,

$$|f(x) - r_{n-1,n}(x)| = \frac{|\psi(x) - S_{n,q}(\psi; x)|}{|x|^{q+1}} = \frac{|\psi^{(q)}(\xi_x) - S_{n,q}^{(q)}(\psi; \xi_x)|}{q!\,|x|}, \quad 0 < \xi_x < |x| \leqslant n^{-2\ell-1},$$

and by (3.6) we get

$$|f(x) - r_{n-1,n}(x)| \leqslant C\frac{\omega(\psi^{(q)}; |x|^{2\ell/(2\ell+1)}/n)}{|x|} \leqslant C|x|^{\sigma(2\ell)/(2\ell+1)-1/(2\ell+1)}n^{1+\sigma}, \tag{3.8}$$

$0 < |x| \leqslant n^{-2\ell-1}$. By using [4, Lemma 1] for (3.7) and working as in [4, p.219] for (3.8), the assertion follows. □

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